

Asymptotic Normality of U -Quantile-Statistics

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Abstract

In 1948, W. Hoeffding introduced a large class of unbiased estimators called U -statistics, defined as the average value of a real-valued m -variate function h calculated at all possible sets of m points from a random sample. In the present paper, we investigate the corresponding robust analogue which we call U -quantile-statistics. We are concerned with the asymptotic behavior of the sample p -quantile of such function h instead of its average. Alternatively, U -quantile-statistics can be viewed as quantile estimators for a certain class of dependent random variables. Examples are given by a slightly modified Hodges-Lehmann estimator of location and the median interpoint distance among random points in space.

Keywords: robust, U -statistics, U -max-statistics, dependent, sample quantile, Hodges-Lehmann

1 Introduction

U -statistics form a very important class of unbiased estimators for distributional properties such as moments or Spearman's rank correlation. A U -statistic of degree m with symmetric kernel h is a function of the form

$$U_n(\xi_1, \dots, \xi_n) = \binom{n}{m}^{-1} \sum_J h(\xi_{i_1}, \dots, \xi_{i_m}), \quad (1.1)$$

where the sum is over $J = \{(i_1, \dots, i_m) : 1 \leq i_1 < \dots < i_m \leq n\}$, ξ_1, \dots, ξ_n are random elements in a measurable space \mathcal{S} , and h is a real-valued Borel function on \mathcal{S}^m , symmetric in its m arguments. In his seminal paper, Hoeffding [5] defined U -statistics for not necessarily symmetric kernels and for

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random points in d -dimensional Euclidean space \mathbb{R}^d . Later the concept was extended to arbitrary measurable spaces. Since 1948, most of the classical asymptotic results for sums of i.i.d. random variables have been formulated in the setting of U -statistics, such as central limit laws, strong laws of large numbers, Berry-Esséen type bounds, and laws of the iterated logarithm.

In this article we replace the average in (1.1) by the sample p th quantile \tilde{H}_{pn} and study its asymptotic distribution. By e.g. replacing the average by the median ($p = 1/2$), ordinary U -statistics are robustified in a natural way.

For any distribution function F , the p th quantile, $0 < p < 1$, is given by

$$\tilde{H}_p = \inf\{x : F(x) \geq p\},$$

which satisfies the inequality

$$F(\tilde{H}_p-) \leq p \leq F(\tilde{H}_p).$$

Here, the sample p th quantile \tilde{H}_{pn} is defined as the p th quantile of the empirical distribution function of the sequence of dependent random variables

$$\{h(\xi_{i_1}, \dots, \xi_{i_m}), \quad 1 \leq i_1 < i_2 < \dots < i_m \leq n\}, \quad (1.2)$$

i.e. \tilde{H}_{pn} is a value that separates the lowest $100p\%$ random variables in (1.2) from the rest.

Under mild smoothness conditions on the distribution function F of $h(\xi_{i_1}, \dots, \xi_{i_m})$, we proof asymptotic normality for this class of estimators for $0 < p < 1$. The exceptions $p = 0$ and $p = 1$, corresponding to the extreme values of the dependent sequence (1.2), were already investigated in Lao and Mayer [6]. For bounded kernels, they established Weibull limit laws for these so called U -max-statistics. Their results are mainly based on a Poisson approximation theorem for U -statistics, see e.g. Barbour et al. [1].

In Section 2 we present the main result of the article and discuss asymptotic relative efficiency of a general U -quantile-statistic with respect to the corresponding ordinary U -statistic. The proof of the main result is shown in Section 3. In Section 4 we apply our results to show asymptotic normality for both a modification and a generalization of the well-known Hodges-Lehmann estimator of location. As a second application, we describe the limiting behavior of the median interpoint distance among a random sample of points in Euclidean space.

2 Asymptotic normality

Asymptotic normality of \tilde{H}_{pn} is stated in the main result of this article.

Theorem 2.1. *Let ξ_1, \dots, ξ_n be i.i.d. \mathcal{S} -valued random elements and $h : \mathcal{S}^m \rightarrow \mathbb{R}$ a symmetric Borel function. Assume that the distribution function F of $h(\xi_1, \dots, \xi_m)$ is continuous at \tilde{H}_p . Left- and right-hand derivatives of F at \tilde{H}_p are denoted by $F'(\tilde{H}_p-)$ and $F'(\tilde{H}_p+)$, respectively, provided they exist. Put*

$$\zeta = \mathbf{P} \left\{ h(\xi_1, \dots, \xi_m) \leq \tilde{H}_p, h(\xi_1, \xi_{m+1}, \dots, \xi_{2m-1}) \leq \tilde{H}_p \right\} - p^2. \quad (2.1)$$

Then, for $0 < p < 1$ and $\zeta > 0$,

(i) *If there exists $F'(\tilde{H}_p-) > 0$, then for $t < 0$,*

$$\lim_{n \rightarrow \infty} \mathbf{P} \left\{ \frac{n^{\frac{1}{2}}(\tilde{H}_{pn} - \tilde{H}_p)}{m\zeta^{\frac{1}{2}}/F'(\tilde{H}_p-)} \leq t \right\} = \Phi(t).$$

(ii) *If there exists $F'(\tilde{H}_p+) > 0$, then for $t > 0$,*

$$\lim_{n \rightarrow \infty} \mathbf{P} \left\{ \frac{n^{\frac{1}{2}}(\tilde{H}_{pn} - \tilde{H}_p)}{m\zeta^{\frac{1}{2}}/F'(\tilde{H}_p+)} \leq t \right\} = \Phi(t).$$

As an immediate consequence of Theorem 2.1, the following result holds.

Corollary 2.2. *If F in Theorem 2.1 possesses a density f in a neighborhood of \tilde{H}_p and f is positive and continuous at \tilde{H}_p , then*

$$\lim_{n \rightarrow \infty} \mathbf{P} \left\{ \frac{n^{\frac{1}{2}}(\tilde{H}_{pn} - \tilde{H}_p)}{m\zeta^{\frac{1}{2}}/f(\tilde{H}_p)} \leq t \right\} = \Phi(t).$$

The continuity assumption for f is required, as otherwise, f could differ from F' on a set with 0 mass.

Remark 1. For $\mathcal{S} = \mathbb{R}$, by conditioning on the common random element ξ_1 , ζ of (2.1) can be written as

$$\zeta = \int \left(\mathbf{P} \left\{ h(x, \xi_2, \dots, \xi_m) \leq \tilde{H}_p \right\} \right)^2 dG(x) - p^2,$$

where G is the distribution function of ξ_1 .

Remark 2. By setting $m = 1$, i.e. if h is a function from \mathcal{S} to \mathbb{R} , Theorem 2.1 implies Theorem A (p. 77) of Serfling [7] on the asymptotic normality of the sample quantiles for i.i.d. random variables.

Remark 3. By the central limit theorem for ordinary U -statistics (see e.g. Hoeffding [5] or Serfling [7]), we are able to compare the asymptotic efficiency of the (robust) U -quantile-statistic (for $p = \frac{1}{2}$) with the asymptotic efficiency of U_n given by (1.1). Assume that the assumptions of Corollary 2.2 are fulfilled. Furthermore, assume that the density f is symmetric about μ , the random variables in (1.2) have finite variance and

$$\zeta_1 = \mathbf{E}((h(\xi_1, \dots, \xi_m) - \mu)(h(\xi_1, \xi_{m+1}, \dots, \xi_{2m-1}) - \mu)) > 0.$$

Then, the ordinary U -statistic U_n based on kernel h is asymptotically normal with variance $m^2\zeta_1$. Hence, by Corollary 2.2,

$$e(\tilde{H}_{pn}, U_n) = f^2(\mu)\zeta_1/\zeta.$$

3 Proof of Theorem 2.1

We follow the proof of asymptotic normality of the usual p th quantile by Serfling (p. 78f) [7], with the necessary adaptations.

Proof of Theorem 2.1. For fixed t write

$$\begin{aligned} G_n(t) &= \mathbf{P} \left\{ \frac{n^{\frac{1}{2}} (\tilde{H}_{pn} - \tilde{H}_p)}{A} \leq t \right\} \\ &= \mathbf{P} \left\{ \tilde{H}_{pn} \leq \tilde{H}_p + tAn^{-\frac{1}{2}} \right\} \\ &= \mathbf{P} \{p \leq U_n(\Delta_{nt})\}, \end{aligned} \tag{3.1}$$

where A is a constant specified later and

$$U_n(\Delta_{nt}) = \binom{n}{m}^{-1} \sum_{i_1 < \dots < i_m} \mathbf{1} \left\{ h(\xi_{i_1}, \dots, \xi_{i_m}) \leq \tilde{H}_p + tAn^{-\frac{1}{2}} \right\}$$

is an ordinary U -statistic with expectation

$$\Delta_{nt} = \mathbf{P} \left\{ h(\xi_{i_1}, \dots, \xi_{i_m}) \leq \tilde{H}_p + tAn^{-\frac{1}{2}} \right\}.$$

By continuity of F at \tilde{H}_p , $\Delta_{nt} \rightarrow p$ as $n \rightarrow \infty$.

Since the kernel of $U_n(\Delta_{nt})$ is either 0 or 1, the third absolute moment λ of $U_n(\Delta_{nt})$ exists. Furthermore, by continuity of probability functions (see e.g. p. 351 in Serfling [7]) and continuity of F at \tilde{H}_p , the quantity

$$\zeta_n =$$

$$\mathbf{P} \left\{ h(\xi_1, \dots, \xi_m) \leq \tilde{H}_p + tAn^{-\frac{1}{2}}, h(\xi_1, \xi_{m+1}, \dots, \xi_{2m-1}) \leq \tilde{H}_p + tAn^{-\frac{1}{2}} \right\} - \Delta_{nt}^2$$

converges to its limit $\zeta > 0$ as $n \rightarrow \infty$. Thus, for the normalized U -statistic

$$U_n^*(\Delta_{nt}) = \frac{n^{\frac{1}{2}} (U_n(\Delta_{nt}) - \Delta_{nt})}{m\zeta_n^{\frac{1}{2}}},$$

by the Berry-Esséen theorem for U -statistics by Callaert and Janson [2],

$$\sup_{t \in \mathbb{R}} |\mathbf{P} \{U_n^*(\Delta_{nt}) \leq t\} - \Phi(t)| \leq \frac{C\lambda}{n^{\frac{1}{2}}m^3\zeta_n^{3/2}} \quad (3.2)$$

holds at least asymptotically as $n \rightarrow \infty$ for an universal constant $0 < C < \infty$.

From (3.1) it follows that

$$\begin{aligned} G_n(t) &= \mathbf{P} \left\{ \frac{n^{\frac{1}{2}} (p - \Delta_{nt})}{m\zeta_n^{\frac{1}{2}}} \leq U_n^*(\Delta_{nt}) \right\} \\ &= \mathbf{P} \{U_n^*(\Delta_{nt}) \geq -c_{nt}\} \end{aligned}$$

with

$$c_{nt} = \frac{n^{\frac{1}{2}} (\Delta_{nt} - p)}{m\zeta_n^{\frac{1}{2}}}.$$

Clearly,

$$\begin{aligned} \Phi(t) - G_n(t) &= \mathbf{P} \{U_n^*(\Delta_{nt}) < -c_{nt}\} - (1 - \Phi(t)) \\ &= \mathbf{P} \{U_n^*(\Delta_{nt}) < -c_{nt}\} - \Phi(-c_{nt}) + \Phi(t) - \Phi(c_{nt}), \end{aligned}$$

and thus, by using the Berry-Esséen bound (3.2),

$$|G_n(t) - \Phi(t)| \leq \frac{C\lambda}{n^{\frac{1}{2}}m\zeta_n^{3/2}} + |\Phi(t) - \Phi(c_{nt})|.$$

The first term on the right hand side vanishes as $n \rightarrow \infty$. It thus remains to show $c_{nt} \rightarrow t$ as $n \rightarrow \infty$. By

$$\begin{aligned} c_{nt} &= \frac{n^{\frac{1}{2}} (\Delta_{nt} - p)}{m\zeta_n^{\frac{1}{2}}} \\ &= \frac{tA}{m\zeta_n^{\frac{1}{2}}} \frac{F(\tilde{H}_p + tAn^{-\frac{1}{2}}) - F(\tilde{H}_p)}{tAn^{-\frac{1}{2}}}, \end{aligned}$$

it follows, for $t > 0$ as $n \rightarrow \infty$,

$$c_{nt} \rightarrow \frac{tAF'(\tilde{H}_p+)}{m\zeta^{\frac{1}{2}}}.$$

Similarly, for $t < 0$ as $n \rightarrow \infty$,

$$c_{nt} \rightarrow \frac{tAF'(\tilde{H}_p-)}{m\zeta^{\frac{1}{2}}}.$$

Choosing

$$A = \frac{m\zeta^{\frac{1}{2}}}{F'(\tilde{H}_p+)}$$

if $t > 0$ and

$$A = \frac{m\zeta^{\frac{1}{2}}}{F'(\tilde{H}_p-)}$$

if $t < 0$, the claimed result follows. \square

4 Examples

4.1 The Hodges-Lehmann estimator of location

As an application of Theorem 2.1 (resp. Corollary 2.2), we deduce asymptotic normality of a slightly modified version of the Hodges-Lehmann estimator [4] of location and of a generalization. The Hodges-Lehmann estimator is given by the median of all Walsh averages

$$\frac{\xi_i + \xi_j}{2}, \quad 1 \leq i \leq j \leq n$$

and estimates the location parameter associated with the one-sample Wilcoxon test, see e.g. Hettmansperger [3].

If the Walsh averages with $i = j$ are dropped from the original definition of the Hodges-Lehmann estimator, this modification can be expressed easily as a U -quantile-statistic with $p = \frac{1}{2}$ and kernel

$$h(x, y) = (x + y)/2.$$

Let ξ_1, \dots, ξ_n be i.i.d. random variables with distribution G and square integrable and continuous density g , symmetric about 0 say, and $g(0) > 0$.

Then $h(\xi_i, \xi_j)$, $1 \leq i < j \leq n$, with common d.f. F have continuous density f and $f(0) > 0$ and thus Corollary 2.2 can be applied directly. Clearly

$$F(z) = \mathbf{P} \left\{ \frac{\xi_i + \xi_j}{2} \leq z \right\} = \int \mathbf{P} \{ \xi_i \leq 2z - x \} g(x) dx.$$

Thus, by symmetry,

$$f(0) = F'(0) = 2 \int g(x)^2 dx.$$

The value of ζ is found easily by Remark 1:

$$\begin{aligned} \zeta + 1/4 &= \int (\mathbf{P} \{ \xi_2 \leq x \})^2 g(x) dx \\ &= \mathbf{E} (G^2(X)) \\ &= \mathbf{E} U^2 \end{aligned}$$

for a standard uniformly distributed random variable U . Thus, $\zeta = 1/3 - 1/4 = 1/12 > 0$.

Corollary 2.2 ensures asymptotic normality with mean 0 and variance σ^2

$$\sigma^2 = \frac{m^2 \zeta}{f^2(0)} = \left(12 \left(\int g^2(x) dx \right)^2 \right)^{-1},$$

which equals the corresponding result for the Hodges-Lehmann estimator, see Hettmansperger [3], p. 37.

In the same way, the asymptotic distributions for U -quantile-statistics with kernels

$$h(x_1, \dots, x_m) = m^{-1} \sum_{i=1}^m x_i, \quad m > 2,$$

can be established by plugging

$$f(0) = m \int \dots \int g(x_1 + \dots + x_{m-1}) g(x_1) \dots g(x_{m-1}) dx_1 \dots dx_{m-1}$$

and

$$\begin{aligned} &\zeta + 1/4 \\ &= \int \left(\int \dots \int G(x_1 + \dots + x_{m-1}) g(x_2) \dots g(x_{m-1}) dx_1 \dots dx_{m-1} \right)^2 g(x_1) dx_1 \end{aligned}$$

into Corollary 2.2.

4.2 Median interpoint distance

A geometric example of a U -quantile-statistic is given by the sample median $\hat{\theta}_n$ of all interpoint distances $\|\xi_i - \xi_j\|$ (with theoretical median θ) of a sample of i.i.d. points ξ_1, \dots, ξ_n with continuous density g in \mathbb{R}^d , $d \geq 1$. Distances are measured with respect to any fixed norm $\|\cdot\|$ on \mathbb{R}^d . The closed unit ball induced by this norm is denoted by \mathbb{B}^d with surface \mathbb{S}^{d-1} and we write

$$\begin{aligned} \{x + \theta\mathbb{B}^d\} &= \{y \in \mathbb{R}^d : \|y - x\| \leq \theta\} \text{ and} \\ \{x + \theta\mathbb{S}^{d-1}\} &= \{y \in \mathbb{R}^d : \|y - x\| = \theta\}. \end{aligned}$$

The asymptotic normal distribution of $\hat{\theta}_n$ is established by Corollary 2.2 and Remark 1. The value of ζ is found via

$$\begin{aligned} \zeta + 1/4 &= \int_{\mathbb{R}^d} (\mathbf{P} \{\|\xi_1 - x\| \leq \theta\})^2 g(x) dx \\ &= \int_{\mathbb{R}^d} (\mathbf{P} \{\xi_1 \in \{x + \theta\mathbb{B}^d\}\})^2 g(x) dx \\ &= \int_{\mathbb{R}^d} \left(\int_{\{x + \theta\mathbb{B}^d\}} g(y) dy \right)^2 g(x) dx, \end{aligned}$$

whereas the density f of the random interpoint distance $\|\xi_1 - \xi_2\|$ at θ is given by

$$\begin{aligned} f(\theta)d\theta &= \mathbf{P} \{\theta \leq \|\xi_1 - \xi_2\| \leq \theta + d\theta\} \\ &= \int_{\mathbb{R}^d} \mathbf{P} \{\theta \leq \|\xi_1 - x\| \leq \theta + d\theta\} g(x) dx \\ &= \int_{\mathbb{R}^d} d\theta \left(\int_{\{x + \theta\mathbb{S}^{d-1}\}} g(y) dy \right) g(x) dx, \end{aligned}$$

hence

$$f(\theta) = \int_{\mathbb{R}^d} \left(\int_{\{x + \theta\mathbb{S}^{d-1}\}} g(y) dy \right) g(x) dx.$$

Corollary 2.2 ensures asymptotic normality.

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